

Semiclassical spinning strings and confining gauge theories

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Abstract

We study multi-charged rotating string states on Type IIB regular backgrounds dual to confining $SU(N)$ gauge theories with (softly broken) $\mathcal{N} = 1$ supersymmetry, in the infra red regime. After exhibiting the classical energy/charge relations for the folded and circular two-charge strings, we compute in the latter case the one loop sigma-model quantum correction. The classical relation has an expansion in positive powers of the analogous of the BMN effective coupling, while the quantum corrections are non perturbative in nature and are not subleading in the limit of infinite charge. We comment about the dual field theory multi-charged hadrons and the implications of our computation for the $AdS/\mathcal{N} = 4$ duality.

1 Introduction

The string/gauge theory correspondence [1] was recently put at work beyond the supergravity approximation of string theory [2, 3]. A crucial role in the game was played by special sets of dual string/gauge theory states parameterized by large quantum numbers. In the master example of the AdS/CFT correspondence, a limit was discovered where the energy of large angular momentum closed string states (rotating along a great circle of S^5) reproduces the conformal dimensions of the corresponding gauge theory operators. The string states in the limit are semiclassical solitons, whose physics is effectively described by string theory on the so called Penrose limit [4] of the $AdS_5 \times S^5$ original background. This results in a parallel plane (pp) wave background [5] on which string theory can be exactly solved [6]. This stimulated the study of more general (multi)spinning string states in $AdS_5 \times S^5$ (see the review [7] and references therein¹) whose simple classical description often provides non trivial predictions on the dual quantum gauge theory.

A lot of work is still required in order to extend the above techniques beyond the master conformal example. The present paper is devoted to the study of multi-spinning string solutions on the regular IIB string backgrounds conjectured to be dual to confining, four dimensional, $SU(N)$ gauge theories². We will concentrate on string solutions relevant for the study of the low energy gauge theory dynamics. In the regime covered by the semiclassical string approximation, the gauge theories we will examine will be coupled to towers of massive (Kaluza-Klein) adjoint fields generically charged under some $U(1)$ symmetry group. The semiclassical closed string solutions describe large (multi)charged massive hadronic states made of these particles.

For all the confining cases present in literature, when the gauge theories are strongly coupled the dual backgrounds asymptotically reach a relatively simple, universal form in the IR. Very roughly, it is a flat $d + 1$ dimensional Minkowski space-time, times a flat q dimensional space, times a transverse $9 - d - q$ sphere (usually those are not really products spaces, but there is some nontrivial mixing between the various factors). The isometries of the sphere correspond to global symmetries for the adjoint Kaluza-Klein matter fields. It is not complicated to look for string solutions on those backgrounds. They should describe gauge theory objects visible in the low energy limit. In the papers [12, 13, 14, 15] *point-like* strings sitting at the origin of the q dimensional space³ and rotating along a great circle of the transverse sphere, were considered. Just as in the $AdS_5 \times S^5$ case, the semiclassical approximation here is effectively described by the string quantization on a generalized Penrose limit of the corresponding original backgrounds. From the gauge theory point of view, the string Hamiltonian describes the non relativistic motion (in a d -dimensional space) and excitations of stringy shaped *hadrons*. Their constituents are the adjoint Kaluza-Klein massive fields. String theory provides a non trivial

¹More recent solutions, not included in the references of [7], can be found in [8].

²Other classical solutions in confining backgrounds were studied in [9, 10, 11].

³This corresponds to the low energy region of the dual gauge theory.

prediction on (the relations between) masses and charges of these hadronic states.

In the present paper we study more general circular and folded⁴ closed string solutions with multiple spins along the transverse sphere in our preferred example (for its simplicity) of the Maldacena-Núñez (MN) [16] solution and its non supersymmetric version (bMN) [17, 18]. The former is conjectured to be dual to an $\mathcal{N} = 1$ $SU(N)$ gauge theory in four dimensions. The latter is dual to the same gauge theory in which supersymmetry is broken by a gluino mass term. In both cases the transverse sphere is an S^3 and so our general solutions will have two spins along it. Analogous solutions for the (softly broken) Klebanov-Strassler⁵ (KS [20] and bKS [21, 22]) case are exactly the same as the (b)MN ones (the relevant background in the far IR is the same) and so we do not present the (b)KS results explicitly. As expected from the symmetries of the problem, the integrable structures playing a crucial role in the example of the AdS/CFT correspondence [23, 24, 7] are manifest in the confining context too. In fact, the trivial but crucial observation that the deep IR geometry of confining backgrounds is of the factorized “flat space \times a sphere” form, makes it evident that the *classical* solutions will be the same as the ones studied in the $AdS/\mathcal{N} = 4$ context, with only a trivial difference in some coefficients. Thus, the classical energy/charge relations for these configurations admit an expansion in positive powers of the effective coupling⁶ λ^2/J^2 . An expansion of this kind is usually called *regular*. It is obtained by taking the large λ, J limit, with λ^2/J^2 fixed, of the classical energy of the string states. The leading term in the limit gives a linear relation between the energy and the total charge of the dual hadrons. We will speculate on a possible interpretation of the regular corrections to this relation as due to the collective binding energy of the hadrons.

The main part of the paper is devoted to the calculation of the quadratic fluctuations around the circular solutions on the bMN background. We will study their quantization and examine the one loop contribution to the zero-point energy. The latter turns out to be non-vanishing for generic values of the supersymmetry breaking parameter and in particular in the supersymmetric case. As a consequence, the energy/charge relations are corrected at the quantum level for these configurations, as it happens in the special case of the point-like, single-charge string [13]. Moreover the one loop corrections are different in nature with respect to the classical ones. In fact, the leading order quantum corrections to the $E \approx J$ relation scale as $1/\lambda$ and thus are not “regular” as the latter. Also it results that, contrarily to what happens in the $AdS_5 \times S^5$ context [25, 26, 7], these quantum corrections (which are anyway subleading with respect to the classical terms because $\lambda \gg 1$), are not subleading in the limit of infinite total charge $J \rightarrow \infty$. As a consequence, the classical result cannot be extrapolated to small

⁴The classical solution for the single charge folded string was presented in [10].

⁵For a review of the (b)MN and KS models see for example [19].

⁶Here $\lambda = e^{\Phi_0} N$, Φ_0 being the value of the dilaton at the origin, and J is the sum of the charges. Note that the power of λ is different from the AdS case because of the presence of the three-form instead of the five-form field strength [12].

values of λ . We will speculate on the possible interpretation of the one-loop string sigma model contributions as non perturbative (in λ) corrections to the masses of the single constituents.

Even if our energy/charge relations cannot be compared with perturbative field theory calculations as in the *AdS/CFT* case, they provide new non-trivial predictions for the mass/charge (Regge-like) relations for the dual hadrons in the strong coupling regime. As pointed out in [15], lattice simulations could in principle provide an “experimental” test for the validity of our results for the string/confining gauge theory correspondence in this sector.

Moreover, our calculation turns out to be relevant for a better understanding of the *AdS/CFT* case too. While in the literature there are plenty of classical solutions, the explicit calculations of the quantum corrections have been done in very few examples [27, 26]. We provide one such computation in a context which is slightly different from *AdS*. This allows us to cast some conclusions on the nature of the one loop correction and its implication for the regularity of the expansion of the energy/charge relation in general backgrounds.

In section 2 we present the (b)MN background in the IR in a form which is useful for the quantization. Then, in section 3, we review the connection of the spinning strings with integrable systems (sec. 3.1) and show the folded (sec. 3.2) and circular (sec. 3.3) string solutions, together with the classical energy/charge relations in both cases. The computation of the one loop correction to this relation for the circular string case is presented in section 4, while in section 5 we give a simple argument for the absence of supersymmetry in our solutions. We discuss the dual multi-charged hadrons in section 6 and conclude with some comments about the implications of our results in section 7. We also include an appendix with the Green-Schwarz superstring on general backgrounds (to second order in the fermions) and one with a brief discussion of the UV finiteness and stability of the circular solution.

2 The (b)MN background

Let us start by briefly reviewing the basic aspects of the MN solution [16] and its softly broken generalization [17, 18]. In the string frame, the MN solution in global coordinates is given by

$$ds_{str}^2 = e^\Phi \left\{ dx_\mu dx^\mu + \alpha' N [d\rho^2 + e^{2g(\rho)} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{4} \sum_a (w^a - A^a)^2] \right\}, \quad (1)$$

$$F^{(3)} = \alpha' N \left[-\frac{1}{4} (w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \frac{1}{4} \sum_a F^a \wedge (w^a - A^a) \right], \quad (2)$$

with a ρ -dependent dilaton whose value at the origin is a continuum parameter Φ_0 and gauge field A given by

$$A = \frac{1}{2} [\sigma^1 a(\rho) d\theta_1 + \sigma^2 a(\rho) \sin \theta_1 d\phi_1 + \sigma^3 \cos \theta_1 d\phi_1]. \quad (3)$$

The one-forms w^a are defined by

$$\frac{i}{2} w^a \sigma^a = dg g^{-1}, \quad g = e^{\frac{i}{2} \psi \sigma_3} e^{\frac{i}{2} \theta_2 \sigma_1} e^{\frac{i}{2} \phi_2 \sigma_3},$$

$$w^1 + iw^2 = e^{-i\psi}(d\theta_2 + i \sin \theta_2 d\phi_2) , \quad w^3 = d\psi + \cos \theta_2 d\phi_2 . \quad (4)$$

The full explicit form of the functions $g(\rho)$, $\Phi(\rho)$, $a(\rho)$ in the supersymmetric case is

$$\begin{aligned} a(\rho) &= \frac{2\rho}{\sinh 2\rho} , \\ e^{2g}(\rho) &= \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4} , \\ e^{2\Phi}(\rho) &= g_s^2 \frac{\sinh 2\rho}{2e^{g(\rho)}} , \end{aligned} \quad (5)$$

where it has to be noticed that what we define as the effective string coupling is really the arbitrary parameter $g_s = e^{\Phi_0}$.

In the softly broken version [17, 18] we know in analytic form only the asymptotic behavior. In the IR it reads

$$\begin{aligned} a(\rho) &= 1 - b\rho^2 + \dots , \\ e^g(\rho) &= \rho - \left(\frac{b^2}{4} + \frac{1}{9}\right)\rho^3 + \dots , \\ \Phi(\rho) &= \Phi_0 + \left(\frac{b^2}{4} + \frac{1}{3}\right)\rho^2 + \dots . \end{aligned} \quad (6)$$

The correspondence with the dual field theory is clear only for $b \in (0, 2/3]$, the value $b = 2/3$ corresponding to the supersymmetric solutions above⁷. In order to explore the IR of the above background it is convenient to perform a gauge transformation on A such that it actually goes to zero when $\rho \rightarrow 0$. This can be done since A , also in the softly broken case, is a pure gauge in the extreme IR [16]. We use the gauge transformation [28] $A \rightarrow h^{-1}Ah + ih^{-1}dh$ with $h = e^{i\sigma^1\theta_1/2}e^{i\sigma^3\phi_1/2}$. The resulting field has the following expression in the $\rho \rightarrow 0$ approximation

$$\begin{aligned} A &= \left(-\frac{b}{2}\rho^2 + \mathcal{O}(\rho^4)\right) \left[\sigma^1(\cos \phi_1 d\theta_1 - \cos \theta_1 \sin \theta_1 \sin \phi_1 d\phi_1) \right. \\ &\quad \left. + \sigma^2(\sin \phi_1 d\theta_1 + \cos \theta_1 \sin \theta_1 \cos \phi_1 d\phi_1) + \sigma^3(\sin^2 \theta_1 d\phi_1) \right] . \end{aligned} \quad (7)$$

In the same regime, we have that

$$e^\Phi = g_s \left[1 + \left(\frac{b^2}{4} + \frac{1}{3}\right)\rho^2 + \mathcal{O}(\rho^4) \right] . \quad (8)$$

Since the coordinates ρ , θ_1 , ϕ_1 are clearly degenerate at $\rho = 0$, it is convenient to use the almost Euclidean coordinates y^a , with $a = 1, 2, 3$, related to ρ, θ_1, ϕ_1 as if these were ordinary

⁷It will be interesting, from the string point of view, to allow b to extend in the full range $[0, 2]$ giving regular solutions [17].

spherical coordinates on \mathbb{R}^3 and the y^a 's the flat ones. Then, in the gauge defined above, we have that

$$\begin{aligned} A^1 &= -b(y^3 dy^1 - y^1 dy^3) , \\ A^2 &= +b(y^2 dy^3 - y^3 dy^2) , \\ A^3 &= -b(y^1 dy^2 - y^2 dy^1) . \end{aligned} \quad (9)$$

Furthermore, the only dimensionful coordinates are the x^μ that parameterize the four flat directions. Then, in order to study the string world-sheet theory on this background, it is more comfortable to rescale these flat coordinates in new dimensionless coordinates $X^\mu = m_0 x^\mu$, where $m_0 = 1/\sqrt{\alpha' N}$ is the mass scale of the glueball and KK fields of the dual SYM theory⁸. In this way, the bMN metric (up to quadratic order in y^a) takes the form

$$\begin{aligned} ds^2 &= \frac{g_s}{m_0^2} \left\{ \left[1 + \left(\frac{b^2}{4} + \frac{1}{3} \right) y^a y^a \right] dX^\mu dX_\mu + dy^a dy^a + \frac{1}{4} \left[1 + \left(\frac{b^2}{4} + \frac{1}{3} \right) y^a y^a \right] w^a w^a + \right. \\ &\quad \left. + \frac{b}{2} \epsilon_{ab2} y^a dy^b w^1 - \frac{b}{2} \epsilon_{ab1} y^a dy^b w^2 + \frac{b}{2} \epsilon_{ab3} y^a dy^b w^3 \right\} . \end{aligned} \quad (10)$$

It is the product of flat 4d Minkowski space-time with an S^3 fibration in a six dimensional manifold. At the point $y^a = 0$ it degenerates into $\mathbb{R}^{3,1} \times S^3$.

Let us now observe that in these coordinates the bMN solution has the killing directions ∂_{ϕ_2} and $\partial_\psi - \partial_{\phi_1}$. In order to make these symmetries manifest, it is convenient to go to a rotated frame in the (y^1, y^2) plane, i.e. we introduce new coordinates Y^a defined by

$$y^1 + iy^2 = e^{-i\psi} (Y^1 + iY^2) , \quad y^3 = Y^3 . \quad (11)$$

It is also useful to introduce the following coordinates on S^3

$$\chi = \frac{\theta_2}{2} , \quad \phi_+ = \frac{1}{2}(\psi + \phi_2) , \quad \phi_- = \frac{1}{2}(\psi - \phi_2) . \quad (12)$$

Then the metric expanded up to the second order in Y^a assumes the following form

$$\begin{aligned} ds^2 &= \frac{g_s}{m_0^2} \left\{ \left[1 + \left(\frac{b^2}{4} + \frac{1}{3} \right) Y^a Y^a \right] dX^\mu dX_\mu + dY^a dY^a + \right. \\ &\quad + \left[1 + \left(\frac{b^2}{4} + \frac{1}{3} \right) Y^a Y^a \right] (d\chi^2 + \sin^2 \chi d\phi_-^2 + \cos^2 \chi d\phi_+^2) + \\ &\quad + [(Y^1)^2 + (Y^2)^2] \left[\left(1 - \frac{b}{2} \right) (d\phi_+ + d\phi_-)^2 - \frac{b}{2} (\cos^2 \chi - \sin^2 \chi) d(\phi_+ + \phi_-) d(\phi_+ - \phi_-) \right] + \\ &\quad + (Y^1 dY^2 - Y^2 dY^1) \left[\left(\frac{b}{2} - 2 \right) (d\phi_+ + d\phi_-) + \frac{b}{2} (\cos^2 \chi - \sin^2 \chi) d(\phi_+ - \phi_-) \right] + \\ &\quad + \frac{b}{2} [-2 \sin \chi \cos \chi Y^1 Y^3 d(\phi_+ + \phi_-) d(\phi_+ - \phi_-) + 2 Y^2 Y^3 d\chi d(\phi_+ + \phi_-) + \end{aligned}$$

⁸We will think to the KK and glueball scale m_0 as the natural scale of the theory. We will work with dimensionless quantities and the right dimensionality will be given by reintroducing m_0 opportunely.

$$+2(Y^3 dY^1 - Y^1 dY^3) d\chi - 2 \sin \chi \cos \chi (Y^2 dY^3 - Y^3 dY^2) d(\phi_+ - \phi_-) \Big\} . \quad (13)$$

In the following we will need the RR 3-form $F^{(3)}$ only at leading order. In the new coordinates it is given by

$$\begin{aligned} F^{(3)} &= \frac{1}{4m_0^2} [-w^1 \wedge w^2 \wedge w^3 + \sum_a F^a \wedge w^a] \\ &= \frac{1}{m_0^2} [2 \sin \chi \cos \chi d\chi \wedge d\phi_- \wedge d\phi_+ - b dY^1 \wedge dY^2 \wedge (\cos^2 \chi d\phi_+ + \sin^2 \chi d\phi_-) + \\ &\quad + b dY^1 \wedge dY^3 \wedge d\chi + b \sin \chi \cos \chi dY^2 \wedge dY^3 \wedge (d\phi_+ - d\phi_-)] . \end{aligned} \quad (14)$$

As can be seen directly from (13) and (14), ∂_{ϕ_+} and ∂_{ϕ_-} are the killing vectors of the internal part of our background. We will use the associated conserved angular momenta J_+ and J_- to classify different classical string solutions and the semiclassical quantization of a subclass of them.

3 Classical string solutions on the (b)MN background

In this paper we will study a class of solutions describing strings sitting at the point $Y^a = 0$ and extending and spinning along the S^3 directions of the transverse space. In particular, we are interested in string configurations having definite angular momenta J_+ and J_- in the internal space. As we will discuss better in section 6, these solutions have a natural dual interpretation in terms of hadronic states made up of KK matter of the dual field theory and generalize the annulons discussed in [12, 13]. Those papers considered the Penrose limit of the background, equivalent to the quadratic expansion around the following point-like solution moving in the internal space

$$X^0 \equiv t = k\tau , \quad \phi_+ = \nu\tau , \quad \chi = \phi_- = 0 , \quad Y^a = X^a = 0 \quad \text{for } a = 1, 2, 3 . \quad (15)$$

The conformal gauge constraint implies that the solution must describe a null geodesic, $\nu^2 - k^2 = 0$. The classical energy and charge of this solution are $E = \lambda k$, $J_+ = \lambda\nu$. Reintroducing the right dimensions using the fundamental scale m_0 , we obtain the pp-wave relation $E = m_0 J_+$.

Let us observe that λ represents both the square of the ratio between the typical radius of curvature of the bMN solution and the fundamental string length, and the ratio between the YM string tension and the square of the KK mass scale m_0^2 . Then the supergravity approximation is valid for $\lambda \gg 1$, i.e. in a regime opposite to the limit $\lambda \ll 1$ in which the KK states decouple from the pure $\mathcal{N} = 1$ SYM. The dependence on the parameter λ in the string world-sheet action can be easily isolated by using the dimensionless metric defined by the rescaling $ds^2 \rightarrow (g_s/m_0^2) ds^2$. Then for example the Nambu-Goto (NG) bosonic string action (obviously, nothing changes in the Polyakov approach) becomes

$$S_B = -\frac{\lambda}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \sqrt{-h} , \quad (16)$$

where $h = \det(h_{\alpha\beta})$ and $h_{\alpha\beta}$ is the induced world-sheet metric.

If \mathcal{X}^m denotes the general ten dimensional spacetime coordinate, we can study the dynamics around any given classical solution $\bar{\mathcal{X}}^m$ by performing an expansion $\mathcal{X}^m = \bar{\mathcal{X}}^m + \lambda^{-1/2}\delta\mathcal{X}^m$. In the supergravity approximation $\lambda \gg 1$ the leading contributions in the world-sheet action come from the action expanded up to the quadratic order in the fluctuating fields, i.e. from the one-loop approximation. Analogously to the case of the collapsed string in $AdS_5 \times S^5$ [3, 27], performing such an expansion around the null geodesic (15) naturally produces the string theories on the pp-wave backgrounds studied in the [12, 13].

In the following sections we present some multispin solutions that generalize the collapsed string solution (15), computing the corresponding generalization of pp-wave relation $E = m_0 J_+$, while the semiclassical analysis of a class of regular circular solutions, analogous to those considered in [25, 26], will be performed in section 4.

3.1 Classical multispin solutions and integrable systems

As said, we will focus on some special class of solutions corresponding to string located at $Y^a = 0$, where the bMN background degenerates into $\mathbb{R}^{3,1} \times S^3$. The latter is true in the (b)KS case too, so everything we will say in the present and following subsections applies to this solution as well. Let us first consider the classical solutions on general grounds, reviewing the discussion presented in [23, 24, 7] and adapting to our case some of the results presented there. To this end, it is sufficient to consider the following effective bosonic action obtained from a truncation of the conformally gauge fixed Polyakov action

$$S = \frac{\lambda}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \left[-\frac{1}{2} \partial^\alpha X^\mu \partial_\alpha X_\mu - \frac{1}{2} \partial^\alpha Z^M \partial_\alpha Z^M + \frac{\Lambda}{2} (Z^M Z^M - 1) \right]. \quad (17)$$

The Euclidean coordinates Z^M , $M = 1, \dots, 4$, constrained by $Z^M Z^M = 1$ (hence the Lagrangian multiplier Λ in the action above), describe the transverse S^3 . Their relation to the (χ, ϕ_-, ϕ_+) coordinates can be given as

$$\begin{aligned} Z^1 &= \sin \chi \cos \phi_- , & Z^2 &= -\sin \chi \sin \phi_- , \\ Z^3 &= \cos \chi \cos \phi_+ , & Z^4 &= \cos \chi \sin \phi_+ . \end{aligned} \quad (18)$$

The action (17) is to be supplemented by the conformal gauge constraints

$$\begin{aligned} \dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu + \dot{Z}^M \dot{Z}^M + Z'^M Z'^M &= 0 , \\ \dot{X}^\mu X'_\mu + \dot{Z}^M Z'^M &= 0 , \\ Z^M Z^M &= 1 . \end{aligned} \quad (19)$$

We will look for rotating string solutions of the form

$$X^0 \equiv t = k\tau , \quad X^a = 0 \quad a = 1, 2, 3 ,$$

$$\begin{aligned} Z^1 + iZ^2 &= r_1(\sigma)e^{iw_1\tau+i\alpha_1(\sigma)} = [z^1(\sigma) + iz^2(\sigma)]e^{iw_1\tau} , \\ Z^3 + iZ^4 &= r_2(\sigma)e^{iw_2\tau+i\alpha_2(\sigma)} = [z^3(\sigma) + iz^4(\sigma)]e^{iw_2\tau} . \end{aligned} \quad (20)$$

From the condition $Z^M Z^M = 1$ we must have $r_1^2 + r_2^2 = 1$, and the periodicity conditions are satisfied if

$$r_i(\sigma + 2\pi) = r_i(\sigma) , \quad \alpha_i(\sigma + 2\pi) = \alpha_i(\sigma) + 2\pi m_i , \quad (21)$$

where $m_i \in \mathbb{Z}$ play the role of winding numbers. The energy and the nonzero spins are

$$E = P^t = \lambda k , \quad J^{12} = \lambda w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} r_1^2(\sigma) , \quad J^{34} = \lambda w_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} r_2^2(\sigma) , \quad (22)$$

where the spins J^{12} and J^{34} satisfy the relation

$$\frac{J^{12}}{\lambda w_1} + \frac{J^{34}}{\lambda w_2} = 1 . \quad (23)$$

All the other charges vanish. It is also useful to associate to any conserved charge Q a rescaled charge \mathcal{Q} defined as

$$Q = \lambda \mathcal{Q} . \quad (24)$$

Substituting the ansatz (20) in the action (17) one obtains the following “effective” one dimensional Lagrangian

$$L = \frac{1}{2}(z'^M z'^M - \omega_M^2 z^M z^M) - \frac{1}{2}\Lambda(z^M z^M - 1) , \quad (25)$$

where $\omega_1 = \omega_2 = w_1$ and $\omega_3 = \omega_4 = w_2$. This action corresponds to the well known $n = 4$ Neumann integrable system. One can get more insights by writing the effective action (25) more explicitly in terms of the variables r_i and α_i

$$L = \frac{1}{2} \sum_{i=1,2} (r_i'^2 + r_i^2 \alpha_i'^2 - w_i^2 r_i^2) - \frac{1}{2}\Lambda(\sum_{i=1,2} r_i^2 - 1) . \quad (26)$$

This action implies immediately that $\alpha_i' = v_i/r_i^2$, where v_i are integrals of motion. We can directly use them by rewriting the effective action as

$$L = \frac{1}{2} \sum_{i=1,2} (r_i'^2 - w_i^2 r_i^2 - \frac{v_i^2}{r_i^2}) - \frac{1}{2}\Lambda(\sum_{i=1,2} r_i^2 - 1) . \quad (27)$$

The conformal gauge constraints (19) imply that

$$k^2 = \sum_{i=1,2} (r_i'^2 + w_i^2 r_i^2 + \frac{v_i^2}{r_i^2}), \quad \sum_{i=1,2} w_i v_i = 0 . \quad (28)$$

From the second of these relations we see that the two integrals of motion v_i are not independent.

As a last step, let us note that we can generalize any given solution $Z^M \equiv \bar{Z}^M(\tau, \sigma)$ of the form (20), considering a more general $X^\mu(\tau, \sigma)$. This class of solutions, which includes the multi-charged generalization of the one describing the usual (i.e. mass squared vs. spin) Regge trajectory [29], is classically given by the standard flat-space mode expansion for the coordinates $X^\mu(\tau, \sigma)$. The only difference with respect to the flat space solution is in the conformal constraint, giving the modified on-shell condition

$$m^2 = -P^\mu P_\mu = \lambda^2 [k^2 + 4 \sum_{n \geq 1} (\alpha_n^\mu \alpha_{-n\mu} + \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n\mu})] , \quad (29)$$

the k factor being given by the internal part of the solution, (20). It would be very interesting to perform a semiclassical analysis of this generalized solution along the lines of section 4, but we postpone this problem to the future.

In the following subsections we will consider two sub-cases of possible solutions of the integrable system (27). The first is obtained by posing $v_i = 0$ and gives the generalization of the point-like solution to multi-spin folded strings. The second family of solutions, presented in subsection 3.3, is constituted by circular solutions analogous to the multi-spin solutions of [25]. Due to the peculiar regularity of these solutions, we choose them as natural candidates to explore the one-loop extension of the classical results. This point will be discussed in section 4.

3.2 Folded solutions

Let us start by considering the case of $v_i = 0$. Then, the effective action (26) reduces to an $n = 2$ Neumann model,

$$L = \frac{1}{2} \sum_{i=1,2} (r_i'^2 - w_i^2 r_i^2) - \frac{1}{2} \Lambda \left(\sum_{i=1,2} r_i^2 - 1 \right) . \quad (30)$$

Instead of giving the general solution of this integrable system, we focus on the simplest one. The condition $v_i = 0$ imply that the angles α_i are constant. We then fix them to zero and from the relations (18) we see that we are looking for solutions of the form

$$\theta_2(\sigma) = \theta_2(\sigma + 2\pi) , \quad t = k\tau , \quad \phi_+ = \nu\tau , \quad \phi_- = \omega\tau , \quad (31)$$

where $\nu = w_2$, $\omega = -w_1$ and we have used $\theta_2 = 2\chi$ instead of χ . The effective action

$$L = (\theta_2')^2 + 2(\omega^2 - \nu^2) \cos \theta_2 , \quad (32)$$

is then equivalent to the classical action for a pendulum in a constant gravitational field, where σ plays the role of the time and $V(\theta) \equiv -2(\omega^2 - \nu^2) \cos \theta_2$ is the potential energy. This system

can be easily integrated using the energy integral of motion e , which is related to k , ω and ν by the conformal constraint

$$e \equiv \theta_2'^2 - 2(\omega^2 - \nu^2) \cos \theta_2 = 4k^2 - 2(\omega^2 + \nu^2) . \quad (33)$$

The charges of these solutions are given by⁹

$$\begin{aligned} E &= \lambda k , \\ J_+ &= \frac{\lambda \nu}{4\pi} \int_0^{2\pi} d\sigma (1 + \cos \theta_2(\sigma)) , \\ J_- &= \frac{\lambda \omega}{4\pi} \int_0^{2\pi} d\sigma (1 - \cos \theta_2(\sigma)) . \end{aligned} \quad (34)$$

From these expressions we can easily recover the general relation

$$E = \frac{k}{\nu} J_+ + \frac{k}{\omega} J_- . \quad (35)$$

We can restrict ourself to the case $\omega^2 \geq \nu^2$ without loosing in generality. In fact the case $\nu^2 \geq \omega^2$ can be obtained from the previous one by means of the simple redefinitions

$$\theta_2 \rightarrow \theta_2 + \pi , \quad \phi_+ \leftrightarrow \phi_- , \quad (36)$$

that correspond to a $\pi/2$ rotation in the planes (3,1) and (2,4) of the \mathbb{R}^4 parameterized by the Z^M .

First of all, let us notice that in the case $\omega^2 = \nu^2$ (absence of gravity), we have the solution

$$\theta_2 = \pm 2\sigma \sqrt{k^2 - \omega^2} + \text{const.} , \quad (37)$$

corresponding to closed circular strings. Taking $k^2 = \omega^2$ we have collapsed strings, for which $E = J_+ - J_- \equiv J$. On the other hand the periodicity condition imposes as other possible choice only $k^2 = 1 + \omega^2$ corresponding to circular strings with energy

$$E = \sqrt{m^2 \lambda^2 + J^2} , \quad (38)$$

being m the number of windings. Considering now the case $\omega^2 > \nu^2$, we must have $e \geq V_{min}$, i.e. $k^2 \geq \nu^2$, and we can consider three kinds of different solutions:

- $e = V_{min}$ i.e. $k^2 = \nu^2$. From (33) we obtain that the only admissible solution is given by $\theta_2(\sigma) \equiv 0$. These kind of solutions correspond to collapsed point-like strings (the pendulum does not oscillate).

⁹In this paper we choose to work with the following sign conventions: $w_1, w_2, m_1 \geq 0$ and $m_2 \leq 0$. As a consequence, we will have $J_+ \geq 0$ and $J_- \leq 0$.

- $V_{min} < e < V_{max}$ i.e. $\nu^2 < k^2 < \omega^2$. The string is folded and arrives at a maximal angle $\theta_2^{max} < \pi$ (the pendulum does oscillate).
- $e = V_{max}$ i.e. $k^2 = \omega^2$. The folded string is at its extreme limit (the pendulum is in the unstable vertical position).
- $e > V_{max}$ i.e. $k^2 > \omega^2$. The string is circular (the pendulum turns around with never vanishing angular velocity).

Since the point-like solution can be considered as a limiting case of the folded one and the circular strings are unstable and will decay into the folded ones by “slipping off the side” [3], we focus our attention on the folded string case where $\nu^2 < k^2 < \omega^2$. The maximal angle $\theta_2^{max} < \pi$ is given by

$$\cos \theta_2^{max} = 1 - \frac{2(k^2 - \nu^2)}{\omega^2 - \nu^2} . \quad (39)$$

From (33) one can write the periodicity condition $\theta_2^{max}(\sigma) = \theta_2^{max}(\sigma + 2\pi)$ in the form

$$\frac{\pi}{2} \sqrt{2(\omega^2 - \nu^2)} = \int_0^{\theta_2^{max}} \frac{d\theta_2}{\sqrt{\cos \theta_2 - \cos \theta_2^{max}}} . \quad (40)$$

From (34), (33) and (40) we can write the angular momenta in the following form

$$\begin{aligned} J_+ &= \frac{\lambda \nu}{\pi \sqrt{2(\omega^2 - \nu^2)}} \int_0^{\theta_2^{max}} d\theta_2 \frac{1 + \cos \theta_2}{\sqrt{\cos \theta_2 - \cos \theta_2^{max}}} , \\ J_- &= \frac{\lambda \omega}{\pi \sqrt{2(\omega^2 - \nu^2)}} \int_0^{\theta_2^{max}} d\theta_2 \frac{1 - \cos \theta_2}{\sqrt{\cos \theta_2 - \cos \theta_2^{max}}} . \end{aligned} \quad (41)$$

If we introduce the new parameter

$$\eta = \frac{1}{2}(1 - \cos \theta_2^{max}) = \frac{k^2 - \nu^2}{\omega^2 - \nu^2} , \quad 0 < \eta < 1 , \quad (42)$$

we can write (40) and (41) in terms of hypergeometric functions

$$\begin{aligned} \sqrt{2(\omega^2 - \nu^2)} &= \sqrt{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; \eta\right) , \\ J_+ &= \lambda \nu \left[1 - \eta + \frac{\eta}{\sqrt{4(\omega^2 - \nu^2)}} F\left(\frac{1}{2}, \frac{1}{2}, 2; \eta\right) \right] , \\ J_- &= \frac{\lambda \omega}{\pi} \left[\eta - \frac{\eta}{\sqrt{4(\omega^2 - \nu^2)}} F\left(\frac{1}{2}, \frac{1}{2}, 2; \eta\right) \right] . \end{aligned} \quad (43)$$

Since it is difficult to extract the general form of the relation $E = E(J_+, J_-)$, we consider the two extreme limit, i.e. short strings ($\eta \ll 1$) and long strings ($1 - \eta \ll 1$).

3.2.1 Short strings

We have to consider the limit $\eta \ll 1$. In this case from (43) and (42) we get

$$\begin{aligned}\omega^2 - \nu^2 &= 1 + \frac{\eta}{2} + O(\eta^2) , \\ k^2 - \nu^2 &= \eta + O(\eta^2) .\end{aligned}\tag{44}$$

From (43), the rescaled angular momenta \mathcal{J}_+ and \mathcal{J}_- defined as in (24) become

$$\begin{aligned}\mathcal{J}_+ &= \nu[1 - \frac{1}{2}\eta + O(\eta^2)] , \\ \mathcal{J}_- &= \frac{1}{2}\omega[\eta + O(\eta^2)] .\end{aligned}\tag{45}$$

Then the relation $\mathcal{E}(\mathcal{J}_+, \mathcal{J}_-)$ assumes the following approximate form

$$\mathcal{E} = \mathcal{J}_+ - \frac{\mathcal{J}_-}{\mathcal{J}_+} \sqrt{1 + \mathcal{J}_+^2} ,\tag{46}$$

giving the BMN-type expansion ($J = J_+ - J_-$)

$$E \sim J - \frac{\lambda^2 J_-}{2J^2} .\tag{47}$$

3.2.2 Long strings

We now consider the limit $1 - \eta \ll 1$. In this case from (43) and (42) we have

$$\begin{aligned}\omega^2 - \nu^2 &= \frac{1}{\pi^2} \ln^2(1 - \eta)[1 - 8 \ln 2 \ln^{-1}(1 - \eta) + o(\ln^{-1}(1 - \eta))] , \\ k^2 - \omega^2 &= o(\ln^{-n}(1 - \eta)) \quad \text{for any } n > 0 ,\end{aligned}\tag{48}$$

while the angular momenta become

$$\begin{aligned}\mathcal{J}_+ &= -2\nu \ln^{-1}(1 - \eta)[1 + 4 \ln 2 \ln^{-1}(1 - \eta) + o(\ln^{-1}(1 - \eta))] , \\ \mathcal{J}_- &= \omega[1 + 2 \ln^{-1}(1 - \eta) + o(\ln^{-1}(1 - \eta))] .\end{aligned}\tag{49}$$

Then, a little bit of algebra results in¹⁰

$$\mathcal{E} = -\mathcal{J}_- \left[1 - \frac{2}{\mathcal{J}_-} \sqrt{\frac{1}{\pi^2} + \frac{1}{4}\mathcal{J}_+^2} + o(\ln^{-1}(1 - \eta)) \right] ,\tag{50}$$

giving the expansion

$$E \sim J + \frac{2\lambda^2}{\pi^2 J_+} .\tag{51}$$

¹⁰This formula reduces to the single charge solution of [10] in the $\mathcal{J}_+ \rightarrow 0$ limit.

3.3 Circular constant radii solutions

In this section we will consider another class of solutions derivable from the general approach presented in section 3.1. These have constant radii and can be considered as the bMN analog of the circular multi-spin string solutions extensively studied in [25]. Again, we follow the general procedure outlined in [23, 24, 7]. Let us now reconsider the effective action (27) with non-zero integrals of motion v_i . The particular class of solutions we are interested in can be obtained by considering the case of constant Λ and correspondingly of constant radii r_i . From the periodicity condition (21) and the relation $\alpha'_i = v_i/r_i^2$ one can easily relate the v_i 's to the m_i 's, i.e. $v_i = m_i a_i^2$. Then, we can always write the solution in the form

$$z^1 + iz^2 = a_1 e^{im_1 \sigma}, \quad z^3 + iz^4 = a_2 e^{im_2 \sigma}, \quad a_1^2 + a_2^2 = 1, \quad (52)$$

and the following relations hold

$$\begin{aligned} w_i^2 &= m_i^2 - \Lambda, & a_1^2 + a_2^2 &= 1, \\ k^2 &= 2(a_1^2 w_1^2 + a_2^2 w_2^2) + \Lambda, & a_1^2 w_1 m_1 + a_2^2 w_2 m_2 &= 0. \end{aligned} \quad (53)$$

The two nonzero spins are

$$J_- = \lambda \mathcal{J}_- = -\lambda a_1^2 w_1, \quad J_+ = \lambda \mathcal{J}_+ = \lambda a_2^2 w_2. \quad (54)$$

From the relations (53) we have that

$$\mathcal{E}^2 = 2 \left(-\mathcal{J}_- \sqrt{m_1^2 - \Lambda} + \mathcal{J}_+ \sqrt{m_2^2 - \Lambda} \right) + \Lambda. \quad (55)$$

The explicit expression of E as a function of (J_-, J_+, m_1, m_2) can be given in approximate form in the limit of very large total spin $\mathcal{J} = \mathcal{J}_+ - \mathcal{J}_-$, as we have done in the folded string case. In fact, from (54) and (53) it follows that

$$\sum_{i=1,2} \frac{|\mathcal{J}_i|}{w_i} = 1 \quad \Rightarrow \quad \sum_{i=1,2} \frac{|\mathcal{J}_i|}{\sqrt{m_i^2 - \Lambda}} = 1. \quad (56)$$

In the large \mathcal{J} limit we have that $\Lambda < 0$ with $|\Lambda| \gg 1$, so that we can make the following approximation

$$\sqrt{|\Lambda|} = \mathcal{J} - \frac{1}{2\mathcal{J}} \left(-m_1^2 \frac{\mathcal{J}_-}{\mathcal{J}} + m_2^2 \frac{\mathcal{J}_+}{\mathcal{J}} \right) + \dots \quad (57)$$

Then, in the limit of large total rescaled spin \mathcal{J} , one obtains the following relation between the energy and the spins

$$E = J \left[1 + \frac{\lambda^2}{2J^2} \left(-m_1^2 \frac{J_-}{J} + m_2^2 \frac{J_+}{J} \right) + \dots \right]. \quad (58)$$

As in the folded string case, this expression is regular in the effective coupling constant λ^2/J^2 . In the following section we will perform a semiclassical analysis of this particular class of solutions. They allow to circumvent some subtleties in the quantization procedure which are present, for example, in the folded string case. As a price to be paid for this, we will see that the solution is not stable for all the values of the parameters, some frequencies being imaginary. But this is expected, the folded string being less energetic than the circular one with the same charges [30]. We envisage that the quantization of the folded strings would give very similar results without instabilities.

4 Semiclassical analysis of the circular solutions

In the coordinates (χ, ϕ_-, ϕ_+) our classical constant radii circular solutions has the following simple form

$$\sin \bar{\chi} = a_1 , \quad \bar{\phi}_- = -(w_1\tau + m_1\sigma) , \quad \bar{\phi}_+ = w_2\tau + m_2\sigma . \quad (59)$$

For $a_1, a_2 \neq 0$ we can focus our attention on the following thin neighborhood around $\chi = \bar{\chi}$ which contains the string world-sheet completely

$$\begin{aligned} \chi &\in (\bar{\chi} - \epsilon, \bar{\chi} + \epsilon) , \\ \phi_- &\in [0, 2\pi] \quad \text{with } \phi_- \simeq \bar{\phi}_- + 2\pi , \\ \phi_+ &\in [0, 2\pi] \quad \text{with } \phi_+ \simeq \bar{\phi}_+ + 2\pi . \end{aligned} \quad (60)$$

This patch has the topology of an interval times a two-torus, with metric

$$ds^2 = \frac{g_s}{m_0^2} [d\chi^2 + \cos^2 \chi d\phi_+^2 + \sin^2 \chi d\phi_-^2] . \quad (61)$$

To gain more insight in our solutions (59), we can make the following linear change of coordinates

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{a_1^2 w_1^2 + a_2^2 w_2^2}} (-a_1^2 w_1 \phi_- + a_2^2 w_2 \phi_+) , \\ v_2 &= \frac{1}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} (-a_1^2 m_1 \phi_- + a_2^2 m_2 \phi_+) . \end{aligned} \quad (62)$$

In the new coordinates (χ, v_1, v_2) , the solutions (59) take the form

$$\begin{aligned} \chi(\tau, \sigma) &\equiv \bar{\chi} \quad \text{with } \sin \bar{\chi} = a_1 , \\ v_1(\tau, \sigma) &\equiv \bar{v}_1(\tau) = \tau \sqrt{a_1^2 w_1^2 + a_2^2 w_2^2} , \\ v_2(\tau, \sigma) &\equiv \bar{v}_2(\sigma) = \sigma \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2} . \end{aligned} \quad (63)$$

4.1 Pull-back of the bMN background on the classical solutions

Let us then consider what is the form of the pull-back of the relevant fields on the solutions we are interested in. The induced metric is conformally flat

$$P_{\text{string}}[ds^2] = h_{\alpha\beta} d\xi^\alpha d\xi^\beta = \frac{g_s}{m_0^2} (a_1^2 m_1^2 + a_2^2 m_2^2) (-d\tau^2 + d\sigma^2) . \quad (64)$$

On the world-sheet, we can make the following choice of the 10d vielbein (the underlined numbers refer to flat directions)

$$\begin{aligned} e^{\underline{0}} &= \frac{\sqrt{g_s}}{m_0 \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} (kdt - \sqrt{a_1^2 w_1^2 + a_2^2 w_2^2} dv_1) , \\ e^{\underline{8}} &= \frac{\sqrt{g_s}}{m_0 \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} (-\sqrt{a_1^2 w_1^2 + a_2^2 w_2^2} dt + kdv_1) , \\ e^{\underline{1,2,3}} &= \sqrt{\frac{g_s}{m_0^2}} dX^{1,2,3} , & e^{\underline{4,5,6}} &= \sqrt{\frac{g_s}{m_0^2}} dY^{1,2,3} , \\ e^{\underline{7}} &= \sqrt{\frac{g_s}{m_0^2}} d\chi , & e^{\underline{9}} &= \sqrt{\frac{g_s}{m_0^2}} dv_2 . \end{aligned} \quad (65)$$

Then, we have that the pull-back of this 10d vielbein on the world-sheet has only the following non-zero elements

$$P_{\text{string}}[e^{\underline{0}}] = \sqrt{\frac{g_s}{m_0^2}} \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2} d\tau , \quad P_{\text{string}}[e^{\underline{9}}] = \sqrt{\frac{g_s}{m_0^2}} \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2} d\sigma . \quad (66)$$

On the world-sheet, the RR field strength takes the form

$$\begin{aligned} F^{(3)} &= -\frac{2}{m_0^2} \left(\frac{m_0^2}{g_s} \right)^{\frac{3}{2}} \frac{a_2 w_2}{a_1 m_1} \left[e^{\underline{0}} \wedge e^{\underline{7}} \wedge e^{\underline{9}} - \sqrt{1 + \left(\frac{a_1 m_1}{a_2 w_2} \right)^2} e^{\underline{7}} \wedge e^{\underline{8}} \wedge e^{\underline{9}} \right] + \\ &+ \frac{b}{m_0^2} \left(\frac{m_0^2}{g_s} \right)^{\frac{3}{2}} \frac{1}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \left[(w_1 + w_2) \frac{a_1^2 m_1}{w_2} e^{\underline{4}} \wedge e^{\underline{5}} \wedge e^{\underline{9}} + \right. \\ &- (m_1 + m_2) \frac{a_2^2 w_2}{m_1} (e^{\underline{0}} \wedge e^{\underline{4}} \wedge e^{\underline{5}} + \sqrt{1 + \left(\frac{a_1 m_1}{a_2 w_2} \right)^2} e^{\underline{4}} \wedge e^{\underline{5}} \wedge e^{\underline{8}}) + \\ &+ a_1 a_2 \left((w_1 + w_2) (e^{\underline{0}} \wedge e^{\underline{5}} \wedge e^{\underline{6}} + \sqrt{1 + \left(\frac{a_1 m_1}{a_2 w_2} \right)^2} e^{\underline{5}} \wedge e^{\underline{6}} \wedge e^{\underline{8}}) + \right. \\ &\left. \left. + (m_1 + m_2) e^{\underline{5}} \wedge e^{\underline{6}} \wedge e^{\underline{9}} \right) \right] + \frac{b}{m_0^2} \left(\frac{m_0^2}{g_s} \right)^{\frac{3}{2}} e^{\underline{4}} \wedge e^{\underline{6}} \wedge e^{\underline{7}} . \end{aligned} \quad (67)$$

To end this section, when we will consider the fermionic sector we will need the explicit form of the non-zero components of the pull-back of the 10d spin connection

$$\Omega_{\tau}^{07} = -a_1 a_2 \frac{m_1^2 - m_2^2}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} ,$$

$$\begin{aligned}
\Omega_{\tau}^{78} &= -a_1 a_2 \frac{m_1^2 - m_2^2}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \sqrt{1 + \frac{a_1^2 m_1^2}{a_2^2 w_2^2}} , \\
\Omega_{\tau}^{79} &= -a_1 a_2 \frac{w_1 m_1 - w_2 m_2}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} , \\
\Omega_{\sigma}^{07} &= -a_1 a_2 \frac{w_1 m_1 - w_2 m_2}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} , \\
\Omega_{\sigma}^{78} &= -a_1 a_2 \frac{w_1 m_1 - w_2 m_2}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \sqrt{1 + \frac{a_1^2 m_1^2}{a_2^2 w_2^2}} , \\
\Omega_{\sigma}^{79} &= -a_1 a_2 \frac{m_1^2 - m_2^2}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} .
\end{aligned} \tag{68}$$

4.2 Bosonic fluctuations

In order to study the quadratic fluctuations around these solutions, we can use the Nambu-Goto formulation¹¹ as well as the Polyakov one. The calculations are very easy in the latter case, due to the simplifications related to the choice of the conformal gauge. We thus consider the following fluctuations of our fields around the solutions (59)

$$\begin{aligned}
t &= k\tau + \frac{\delta t}{\sqrt{\lambda}} , & \phi_- &= \bar{\phi}_- + \frac{\delta \phi_-}{\sqrt{\lambda}} , & \phi_+ &= \bar{\phi}_+ + \frac{\delta \phi_+}{\sqrt{\lambda}} , \\
\chi &= \bar{\chi} + \frac{\delta \chi}{\sqrt{\lambda}} , & X^a &= \frac{\delta X^a}{\sqrt{\lambda}} , & Y^a &= \frac{\delta Y^a}{\sqrt{\lambda}} .
\end{aligned} \tag{69}$$

In order to write the Lagrangian in a simple form, it is convenient to introduce new coordinates, inspired by the vielbein (65),

$$\begin{aligned}
\delta \tilde{t} &= \frac{1}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \left(k \delta t - \delta v_1 \sqrt{a_1^2 w_1^2 + a_2^2 w_2^2} \right) , \\
\delta \tilde{v}_1 &= \frac{1}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \left(-\delta t \sqrt{a_1^2 w_1^2 + a_2^2 w_2^2} + k \delta v_1 \right) .
\end{aligned} \tag{70}$$

In the conformal gauge in which the world-sheet metric is fixed to be equal to the induced metric (64), the bosonic part of the Polyakov action is given by

$$\begin{aligned}
S_P &= -\frac{\lambda}{4\pi} \int d\tau \int_0^{2\pi} d\sigma (-h_{\tau\tau} + h_{\sigma\sigma}) = \\
&= -\frac{\lambda}{4\pi} \int d\tau \int_0^{2\pi} d\sigma [(-h_{\tau\tau} + h_{\sigma\sigma})| + (-\delta h_{\tau\tau} + \delta h_{\sigma\sigma})| + \frac{1}{2}(-\delta^2 h_{\tau\tau} + \delta^2 h_{\sigma\sigma})| + \dots] .
\end{aligned} \tag{71}$$

The conformal constraints $h_{\tau\tau} + h_{\sigma\sigma} = 0$, $h_{\tau\sigma} = 0$ at leading order read

$$-\partial_{\tau} \delta \tilde{t} + \partial_{\sigma} \delta v_1 + \frac{2a_1 a_2 (m_1^2 - m_2^2)}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \delta \chi = 0 ,$$

¹¹In this case, thanks to the regularity of our solutions (the induced metric is conformally flat with constant conformal factor) the usual problems related to this formulation are not present.

$$\partial_\tau \delta v_2 - \partial_\sigma \delta \tilde{t} + \frac{2a_1 a_2 (w_1 m_1 - w_2 m_2)}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \delta \chi = 0 . \quad (72)$$

After imposing these constraints to the Polyakov action we get the same result we would have obtained in the Nambu-Goto formulation. The fields $\delta \tilde{t}$ and δv_2 decouple completely from the action and represent unphysical degrees of freedom, and the bosonic action for the remaining eight transverse fluctuation is found, after some algebra, to be equal to

$$\begin{aligned} S^{fl} = & -\frac{1}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \left\{ \eta^{\alpha\beta} \partial_\alpha \delta X^a \partial_\beta \delta X^a + \eta^{\alpha\beta} \partial_\alpha \delta Y^a \partial_\beta \delta Y^a + \eta^{\alpha\beta} \partial_\alpha \delta \chi \partial_\beta \delta \chi + \right. \\ & + \eta^{\alpha\beta} \partial_\alpha \delta \tilde{v}_1 \partial_\beta \delta \tilde{v}_1 + A(\delta \chi)^2 + B\delta \chi [(m_2^2 - m_1^2) \partial_\tau \delta \tilde{v}_1 + (w_1 m_1 - w_2 m_2) \partial_\sigma \delta \tilde{v}_1] + \\ & + M \delta Y^a \delta Y^a + ((\delta Y^1)^2 + (\delta Y^2)^2) M_1 + (\delta Y^1 \partial_\tau \delta Y^2 - \delta Y^2 \partial_\tau \delta Y^1) M_2 + \\ & + (\delta Y^1 \partial_\sigma \delta Y^2 - \delta Y^2 \partial_\sigma \delta Y^1) M_3 + (\delta Y^2 \partial_\tau \delta Y^3 - \delta Y^3 \partial_\tau \delta Y^2) M_4 + \\ & \left. + (\delta Y^2 \partial_\sigma \delta Y^3 - \delta Y^3 \partial_\sigma \delta Y^2) M_5 \right\} , \quad (73) \end{aligned}$$

where

$$\begin{aligned} A &= -4 \frac{a_1^2 a_2^2}{a_1^2 m_1^2 + a_2^2 m_2^2} [(m_1^2 - m_2^2)^2 - (w_1 m_1 - w_2 m_2)^2] , \\ B &= \frac{4k a_1 a_2}{\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2} \sqrt{a_1^2 w_1^2 + a_2^2 w_2^2}} , \\ M &= 2 \left(\frac{b^2}{4} + \frac{1}{3} \right) (a_1^2 m_1^2 + a_2^2 m_2^2) , \\ M_1 &= \left(1 - \frac{b}{2} \right) [(m_1 - m_2)^2 - (w_1 - w_2)^2] , \\ M_2 &= \left(2 - \frac{b}{2} \right) (w_2 - w_1) - \frac{b}{2} (a_2^2 - a_1^2) (w_1 + w_2) , \\ M_3 &= \left(-2 + \frac{b}{2} \right) (m_2 - m_1) + \frac{b}{2} (a_2^2 - a_1^2) (m_1 + m_2) , \\ M_4 &= b a_1 a_2 (w_2 + w_1) , \\ M_5 &= -b a_1 a_2 (m_2 + m_1) . \quad (74) \end{aligned}$$

4.3 The fermionic sector

The term of the superstring action quadratic in the fermions is given by (see also Appendix A)

$$S_{(\text{ferm})} = \frac{i}{4\pi\alpha'} \int d\tau \int_0^{2\pi} d\sigma [\sqrt{-h} h^{\alpha\beta} \delta^{IJ} - \epsilon^{\alpha\beta} (\sigma_3)^{IJ}] \bar{\theta}^I \rho_\alpha (D_\beta \theta)^J , \quad (75)$$

where $h_{\alpha\beta}$ is the induced metric, $\rho_\alpha = \partial_\alpha x^m e_m^a \Gamma_a$ and D_α is the pullback on the world-sheet of

$$\begin{aligned} D_m &= \nabla_m + \frac{1}{8 \cdot 3!} e^\Phi F_{abc}^{(3)} \Gamma_m^{abc} \sigma_1 , \\ \nabla_m &= \partial_m + \frac{1}{4} \Omega_{mab} \Gamma^{ab} . \quad (76) \end{aligned}$$

After imposing the κ -symmetry gauge

$$\theta^1 = \theta^2 = \theta , \quad (77)$$

and performing the usual rescalings in order to factorize the dependence on λ

$$ds^2 \rightarrow \frac{g_s}{m_0^2} ds^2 , \quad e^a \rightarrow \left(\frac{g_s}{m_0^2} \right)^{\frac{1}{2}} e^a , \quad \theta \rightarrow \left(\frac{g_s}{m_0^2} \right)^{\frac{1}{4}} \frac{\theta}{\sqrt{\lambda}} , \quad (78)$$

a straightforward calculation shows that the fermionic string action (75) takes the form

$$\begin{aligned} S_{(\text{ferm})} = & \frac{i}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \bar{\theta} \left\{ \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2} (-\Gamma_0 \partial_\tau + \Gamma_9 \partial_\sigma) + \right. \\ & + \frac{a_1 a_2}{2} \left[\sqrt{1 + \frac{a_1^2 m_1^2}{a_2^2 w_2^2}} \left((m_1^2 - m_2^2) \Gamma_{\underline{078}} - (w_1 m_1 - w_2 m_2) \Gamma_{\underline{789}} \right) + (w_1 m_2 - w_2 m_1) \Gamma_{\underline{079}} \right] \\ & + \frac{b}{4} \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2} \left[\sqrt{1 + \left(\frac{a_1 m_1}{a_2 w_2} \right)^2} \left((m_1 + m_2) \frac{a_2^2 w_2}{m_1} \Gamma_{\underline{458}} - a_1 a_2 (w_1 + w_2) \Gamma_{\underline{568}} \right) + \right. \\ & \left. \left. - \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2} \Gamma_{\underline{467}} \right] \right\} \theta . \quad (79) \end{aligned}$$

4.4 One loop correction to the energy

In order to calculate the one-loop correction to the energy/charge relation (58) in the large k limit, we will approximate the series over n of the frequencies with integrals¹² in $x \equiv n/k$ [27, 32, 31, 26]. We will then derive the characteristic frequencies for the string fields in the large k , fixed x limit. The form of the metric (13), where the charges J_+ , J_- correspond to Killing directions, makes it straightforward to show that, since for our solutions $t = k\tau$, the world-sheet vacuum energy we are going to calculate corresponds to k times the leading quantum correction to the space-time energy/charge relation (see appendix A of [27]). In appendix B we show that the theory is UV finite and briefly discuss its stability.

The action (73) accounts for the eight physical bosonic fields. Three of them, the δX^a ones, are massless as expected. Their frequencies are therefore $\omega_b^i = n = kx$, $i = 1, 2, 3$.

¹²Let us note that in literature one often finds another way of calculating the zero-point energy. It is based on *renormalizing* à la Casimir the bosonic and fermionic series separately, and eventually sum the two contributions. The renormalization amounts in subtracting to the series exactly their integral approximation (the one we are going to calculate). Obviously, since the latter is the dominant contribution at large k , using this procedure one finds that in this limit the corrections are always exponentially vanishing, by construction. This would be true for *every* semiclassical solution studied in this context. As such, the discussion about the subleading behavior, at large k , of the quantum corrections would be trivial, the quantum contributions being subleading by construction. But, since there is no convincing argument in favor of this renormalization procedure (see the discussions in [31] and appendix D of [15]), we will not use it and we will simply calculate the (finite) vacuum energy approximating it with integrals.

Diagonalizing the equations of motion of the $\delta\tilde{v}_1$, $\delta\chi$ fields we find the equation for the frequencies

$$(2\omega_b^2 - 2n^2 - A)^2 - A^2 - B^2[(m_2^2 - m_1^2)\omega_b + (w_1m_1 - w_2m_2)n]^2 = 0 . \quad (80)$$

In the large k , fixed x limit the two frequencies have then the form

$$\omega_b \sim k(\sqrt{x^2 + 1} \pm 1) \pm \frac{|x|(m_1 + m_2)}{\sqrt{x^2 + 1}} + O\left(\frac{1}{k}\right) . \quad (81)$$

For all the bosonic and fermionic frequencies we do not write down the (involved) expressions of the k^{-1} coefficients because we will not need them in the following.

Diagonalizing the equations of motion of the δY^a fields we get the following equation

$$(\omega_b^2 - n^2 - M - M_1)^2 = (nM_5 + \omega_b M_4)^2 + (nM_3 + \omega_b M_2)^2 - \frac{M_1(nM_5 + \omega_b M_4)^2}{\omega_b^2 - n^2 - M} . \quad (82)$$

The large k , fixed x limits of the three frequencies read

$$\begin{aligned} \omega_b &\sim k\left(\sqrt{x^2 + \frac{b^2}{4}} \pm \frac{b}{2}\right) \pm \frac{|x|(b/4 - 1)(m_1 + m_2)}{\sqrt{x^2 + \frac{b^2}{4}}} + O\left(\frac{1}{k}\right) , \\ \omega_b &\sim k|x| + O\left(\frac{1}{k}\right) . \end{aligned} \quad (83)$$

Thus, the sum of the bosonic frequencies in the limit gives¹³

$$\sum \omega_b = k\left(4|x| + 2\sqrt{x^2 + \frac{b^2}{4}} + 2\sqrt{x^2 + 1}\right) + O\left(\frac{1}{k}\right) . \quad (84)$$

Remarkably, this is exactly the expression one finds in the Penrose limit calculation for the bMN background [13]. The frequencies are slightly different but retain the main features: we have three massless world-sheet bosons describing the flat three special directions, two modes¹⁴ (the “universal sector” v bosons in the notation of [12, 13]) which describe the three-sphere excitations, and two b -dependent modes (the u bosons [12, 13]) which, together with a fourth massless mode (the z field [12, 13] whose masslessness is peculiar of the (b)MN background), describe the fibering \mathbb{R}^3 geometry. This pattern of modes is universal in the Penrose limit of confining backgrounds [15].

The equation of motion for the fermions coming from the action (79) is

$$[(-\Gamma_0\partial_\tau + \Gamma_9\partial_\sigma) + C\Gamma_{\underline{078}} + D\Gamma_{\underline{789}} + E\Gamma_{\underline{079}} + F\Gamma_{\underline{458}} + G\Gamma_{\underline{568}} + H\Gamma_{\underline{467}}]\theta = 0 , \quad (85)$$

¹³Note that the $O(k^0)$ term vanishes.

¹⁴Remember that we rescaled all our quantities factorizing out m_0 , which is the unit measure for the masses. Reestablishing it simply amounts in the rescaling $k \rightarrow k/m_0$.

where

$$\begin{aligned}
C &= \frac{a_1 a_2}{2\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \sqrt{1 + \frac{a_1^2 m_1^2}{a_2^2 w_2^2} (m_1^2 - m_2^2)} , \\
D &= -\frac{a_1 a_2}{2\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} \sqrt{1 + \frac{a_1^2 m_1^2}{a_2^2 w_2^2} (w_1 m_1 - w_2 m_2)} , \\
E &= \frac{a_1 a_2}{2\sqrt{a_1^2 m_1^2 + a_2^2 m_2^2}} (w_1 m_2 - w_2 m_1) , \\
F &= \frac{b}{4} \sqrt{1 + \left(\frac{a_1 m_1}{a_2 w_2}\right)^2} (m_1 + m_2) \frac{a_2^2 w_2}{m_1} , \\
G &= -\frac{b}{4} \sqrt{1 + \left(\frac{a_1 m_1}{a_2 w_2}\right)^2} a_1 a_2 (w_1 + w_2) , \\
H &= -\frac{b}{4} \sqrt{a_1^2 m_1^2 + a_2^2 m_2^2} .
\end{aligned} \tag{86}$$

Applying to the right of (85) the operator

$$(\Gamma_{\underline{9}} \partial_\tau - \Gamma_{\underline{0}} \partial_\sigma) + C \Gamma_{\underline{978}} + D \Gamma_{\underline{078}} + E \Gamma_{\underline{7}} + F \Gamma_{\underline{45809}} + G \Gamma_{\underline{56809}} + H \Gamma_{\underline{46709}} , \tag{87}$$

we get the equation

$$\begin{aligned}
& [-\partial_\tau^2 + \partial_\sigma^2 + 2(C\partial_\sigma + D\partial_\tau) \Gamma_{\underline{0978}} - C^2 + D^2 - E^2 - F^2 - G^2 - H^2 + \\
& \quad - 2EF \Gamma_{\underline{097845}} - 2CG \Gamma_{\underline{097856}} + 2FH \Gamma_{\underline{5678}} - 2GH \Gamma_{\underline{4578}}] \theta = 0 .
\end{aligned} \tag{88}$$

Choosing the four dimensional Weyl representation for the gamma matrices $\Gamma_{\underline{0,9,7,8}}$, for which $\Gamma_{\underline{0978}} = \text{diag}(-i, -i, i, i)$ and $\Gamma_{\underline{78}} = \text{diag}(-i, i, -i, i)$, and the two dimensional $\Gamma_{\underline{56}} = \text{diag}(-i, i)$, $\Gamma_{\underline{45}} = \text{antidiag}(-i, -i)$, the equation for the eight frequencies finally read

$$\begin{aligned}
& (\omega_f^2 - n^2 - C^2 + D^2 - E^2 - F^2 - G^2 - H^2 + 2Cn + 2D\omega_f)^2 \\
& \quad - 4(EF + GH)^2 - 4(-EG + FH)^2 = 0 , \\
& (\omega_f^2 - n^2 - C^2 + D^2 - E^2 - F^2 - G^2 - H^2 + 2Cn + 2D\omega_f)^2 \\
& \quad - 4(EF - GH)^2 - 4(EG + FH)^2 = 0 , \\
& (\omega_f^2 - n^2 - C^2 + D^2 - E^2 - F^2 - G^2 - H^2 - 2Cn - 2D\omega_f)^2 \\
& \quad - 4(EF - GH)^2 - 4(EG + FH)^2 = 0 , \\
& (\omega_f^2 - n^2 - C^2 + D^2 - E^2 - F^2 - G^2 - H^2 - 2Cn - 2D\omega_f)^2 \\
& \quad - 4(EF + GH)^2 - 4(EG - FH)^2 = 0 .
\end{aligned} \tag{89}$$

The large k , fixed x limit for the first four frequencies is

$$\omega_f \sim k \left[\frac{1}{2} + \sqrt{\left(\frac{1}{2} \pm \frac{b}{4}\right)^2 + x^2} \right] - \frac{|x|(m_1 + m_2)}{2\sqrt{\left(\frac{1}{2} \pm \frac{b}{4}\right)^2 + x^2}} + O\left(\frac{1}{k}\right) , \tag{90}$$

while for the last four it reads

$$\omega_f \sim k \left[-\frac{1}{2} + \sqrt{\left(\frac{1}{2} \pm \frac{b}{4}\right)^2 + x^2} \right] + \frac{|x|(m_1 + m_2)}{\sqrt{\left(\frac{1}{2} \pm \frac{b}{4}\right)^2 + x^2}} + O\left(\frac{1}{k}\right). \quad (91)$$

The sum of the fermionic frequencies is then¹⁵

$$\sum \omega_f = k \left[4\sqrt{\left(\frac{1}{2} + \frac{b}{4}\right)^2 + x^2} + 4\sqrt{\left(\frac{1}{2} - \frac{b}{4}\right)^2 + x^2} \right] + O\left(\frac{1}{k}\right), \quad (92)$$

which is *not* equal to the bosonic one (84) already at level k . Again, even if the single frequencies are different from the pp-wave ones [13], their sum is exactly the same.

As a result, the leading one-loop sigma-model contribution to the string zero-point energy (and so to the energy/charge relation) in the large k limit is non vanishing

$$\begin{aligned} E_1 &= \frac{1}{2k} \sum_{n=-\infty}^{\infty} \left[\sum_b \omega_b(n) - \sum_f \omega_f(n) \right] \\ &\approx -\frac{k}{2} \left[\frac{b^2}{4} \log \frac{b^2}{4} - 2\left(\frac{1}{2} + \frac{b}{4}\right)^2 \log \left(\frac{1}{2} + \frac{b}{4}\right)^2 - 2\left(\frac{1}{2} - \frac{b}{4}\right)^2 \log \left(\frac{1}{2} - \frac{b}{4}\right)^2 \right], \end{aligned} \quad (93)$$

where in the last step we have approximated the series with integrals. It is of course the same zero-point energy of the Penrose limit of the bMN theory in the large charge limit [13]. Calling m_b and m_f the masses as they appear in the sum of the frequencies (for example, we have for the bosons $\sqrt{x^2 + m_b^2}$ with $m_b = (0, 1, b/2)$), the zero point energy (93) has the “universal” form

$$E = -\frac{k}{4} \left[\sum_b m_b^2 \log m_b^2 - \sum_f m_f^2 \log m_f^2 \right]. \quad (94)$$

It is exactly the same form one finds in the large class of plane-wave backgrounds studied in [31]. Moreover, the finiteness of E_1 is guaranteed by the relation

$$\sum_b m_b^2 = \sum_f m_f^2. \quad (95)$$

In the Penrose limit case [12, 13, 14, 15] and on $AdS_5 \times S^5$ [26] it is a consequence of the supergravity field equations. In our case, it can be viewed as the indication that the quantization procedure we adopted is consistent.

Let us comment on the relation (93) in some detail. First we notice that the one-loop contribution to the zero-point energy is linear in k and it is always *negative* for $b \in (0, 2/3]$ [31, 13]. The fact that the circular string gives the same one-loop correction as the point-like string of the Penrose limit can be understood as follows. In section 4 we argued that the circular

¹⁵Again, the $O(k^0)$ term vanishes.

string is wrapped on a two-torus, with every point of the string moving orthogonally to the string displacement. Now, the boost of a single point is exactly the Penrose limit theory, having the zero-point energy (93). Since, as can be seen from (22), for our case of constant radii the angular momentum density is constant along the string, the total zero-point energy is nothing than the average of the same E_1 for every point of the string, i.e. E_1 itself.

The very same reasoning can be applied to the circular string configurations on the five-sphere of $AdS_5 \times S^5$. In that case too one obtains an average of the Penrose limit theory for every point of the string to leading order in the large k expansion. The difference is that on this plane-wave the zero-point energy is vanishing, due to the underlying supersymmetry, and so the order k one-loop correction is zero for the circular strings.

Curiously, although outside the range of the values for which the correspondence with the field theory is sensible, the value $b = 2$ gives a zero for E_1 . This value of b corresponds to a regular geometry which interpolates between a b -deformed MN background for small radius and a factorized geometry $\mathbb{R}^4 \times \mathbb{R}^3 \times S^3$, with unit sphere, for large radius [17] (in the notation of that paper this critical value is $b = 1/2$). Again, in the large k limit, the zero-point energy at order k is zero because the zero-point energy on the pp-wave is vanishing in this theory. The latter property is due to the fact that on these pp-wave backgrounds [12, 13] the world-sheet theory becomes supersymmetric precisely for $b = 2$, as can be seen from the spectrum [13].

5 Supersymmetry analysis

The four supersymmetries preserved by the MN background are broken by our string solitons. This can be understood in the following way. Suppose that ϵ is the killing spinor of the MN solution. It satisfies the Killing spinor equations

$$D_m \epsilon = 0, \quad \Delta \epsilon = 0, \quad (96)$$

where D_m and Δ are the operators that enter the supersymmetry transformation of the gravitino and the dilatino, whose explicit form is given in appendix A. We are using double spinor notation, i.e. $\epsilon = (\epsilon^1, \epsilon^2)$. Then, using the decompositions (107) and (111) and posing

$$\begin{aligned} D_m^{(0)} &= \hat{D}_m^{(0)} \otimes \mathbf{1}_{2 \times 2}, & W_m &= \hat{W}_m \otimes \sigma_1, \\ \Delta^{(1)} &= \hat{\Delta}^{(1)} \otimes \mathbf{1}_{2 \times 2}, & \Delta^{(2)} &= \hat{\Delta}^{(2)} \otimes \sigma_1, \end{aligned} \quad (97)$$

we can make explicit the killing equation in the following form

$$\begin{aligned} \hat{D}_m^{(0)} \epsilon^1 + \hat{W}_m \epsilon^2 &= 0, & \hat{D}_m^{(0)} \epsilon^2 + \hat{W}_m \epsilon^1 &= 0, \\ \hat{\Delta}^{(1)} \epsilon^1 + \hat{\Delta}^{(2)} \epsilon^2 &= 0, & \hat{\Delta}^{(1)} \epsilon^2 + \hat{\Delta}^{(2)} \epsilon^1 &= 0. \end{aligned} \quad (98)$$

Due to the Poincare isometry group of the four flat directions, these conditions must have a unique solution up to a Lorentz transformation that permits to obtain the four independent

killing spinors. From (98) it is clear that if $\epsilon = (\epsilon^1, \epsilon^2)$ is a killing spinor, then $\tilde{\epsilon} = (\epsilon^2, \epsilon^1)$ is a killing spinor too. But the second cannot be in general obtained from the first one by a Lorentz transformation unless ϵ^1 and ϵ^2 are proportional¹⁶. We then conclude that

$$\epsilon^1 \propto \epsilon^2 . \quad (99)$$

This is indeed consistent with the hypothesis $\epsilon^1 = \epsilon^2$ made in [33] in the calculation of the explicit form of the supersymmetries of the MN solution. Let us now consider the GS superstring action on the MN background, whose general form is given in appendix A. The chiral operator appearing in the kappa-symmetry transformation of the GS superstring is of the form (see appendix A)

$$\Gamma_{F1} = \frac{\epsilon^{\alpha\beta}}{2\sqrt{-h}} \Gamma_{\alpha\beta} \sigma_3 = \hat{\Gamma}_{F1} \otimes \sigma_3 . \quad (100)$$

Then, a (nondegenerate) string configuration preserve the MN supersymmetry ϵ if $(1 + \Gamma_{F1})\epsilon = 0$, or more explicitly

$$(1 + \hat{\Gamma}_{F1})\epsilon^1 = 0 , \quad (1 - \hat{\Gamma}_{F1})\epsilon^2 = 0 . \quad (101)$$

Since $(1 \pm \hat{\Gamma}_{F1})$ are orthogonal projections, the conditions (99) and (101) cannot be contemporaneously satisfied. In the point-like degenerate case supersymmetry breaking is still present [12]: even if the pp-wave background preserves sixteen supersymmetries, these do not give a supersymmetric spectrum, since they are not linearly realized. As we have seen in the previous sections, in the large k limit the circular solution behaves like a bunch of point-like, non-interacting strings. As a consequence, supersymmetry is not restored in this limit. In the case of $b = 2$ the supersymmetry *is* linearly realized on the pp-wave, and it would be very interesting to explicitly verify if this is the case also in the $J \rightarrow \infty$ limit of our circular strings¹⁷.

6 Multicharged hadrons in the dual picture

Let us now discuss the hadrons dual to the string solutions we have found. We have analyzed configurations of spinning strings on the internal three-sphere, carrying two charges. These are generalizations of the point-like string, carrying a single charge, studied in the Penrose limit in [12] for the MN case and in [13] for the bMN one. In the latter paper, an interpretation in the dual field theory for some string modes were given. We will now extend that discussion to the two-charge case.

¹⁶This can be understood by choosing the four dimensional Weyl representation for the killing spinors. Then, in two-components formalism, the general 4d Lorentz transformation is realized by the action of an unimodular 2×2 complex matrix $M \in SL(2, \mathbb{C})$. Taking $\epsilon^1 = (1, 0)$ and imposing $M^2 \epsilon^1 = \epsilon^1 = 1$ with $M \in SL(2, \mathbb{C})$, it is simple to see that $\epsilon^2 = M \epsilon^1 = \pm \epsilon^1$.

¹⁷We thank T. Mateos for a discussion on this issue.

The dual field theory is a four dimensional (softly broken for the bMN case) (S)YM coupled to a (infinite) tower of KK fields [16]. It comes from the world-volume low energy theory of five-branes upon wrapping the latter on a two-sphere inside a (non compact) Calabi-Yau manifold. The latter condition translates in the field theory in a twisting in the two-sphere directions, an operation which allows to retain $\mathcal{N} = 1$ supersymmetry (eventually broken).

Among all the fields in the theory, the lightest KK scalars are the interesting ones for our purpose¹⁸, since the gauge degrees of freedom are uncharged under the internal symmetries. They come from the four scalars of the five-brane theory parameterizing the four directions transverse to the branes in flat space. Upon wrapping of the branes, these scalars get all the same mass, proportional to m_0 . Moreover, being charged under the R-symmetry $SO(4)$ before wrapping, after wrapping and twisting they are charged under the two surviving $U(1)_{l,r}$ inside $SU(2)_{l,r}$ from $SO(4) \sim SU(2)_l \times SU(2)_r$. The combinations of these two $U(1)$'s are exactly the ones corresponding to our two charges.

It is not difficult to see that the scalars are in the bifundamental of the two groups, so that we can name them according to these two charges as $A_{\pm,\pm}$, the first (second) entry referring¹⁹ to $U(1)_l$ ($U(1)_r$) [13]. In the single charge $J_+ = J_l + J_r$ case studied in [12, 13], the state of the form $\text{Tr}(A_{+-})^p |0\rangle_{FT}$, $p \gg 1$, is identified with the string ground state. In fact, since each component has charge $J_+ = 1$ and mass²⁰ m_0 , the energy/charge relation reads $E = m_0 J_+$. This corresponds to the energy of the string ground state $H = E - m_0 J_+ = 0$, up to the zero-point energy correction. The origin of the latter is unclear in field theory. For some considerations, we refer to [32, 13, 15, 34] and to the end of the present discussion. Let us stress that the duality we are talking about is *not* between string states and operators, rather it is a correspondence between string states and *hadrons*, named “annulons” in [12].

In the two-charge case we are considering in this paper, together with the charge J_+ , under which the scalar A_{-+} has charge -1 and the other two scalars A_{++} , A_{--} are uncharged, we have the charge $J_- = J_r - J_l$, under which A_{+-} , A_{-+} are uncharged, while A_{++} , A_{--} have charges -1 , $+1$ respectively. Thus, we immediately conclude that the string ground states are dual to hadrons which we will symbolically write as $\text{Tr}[(A_{+-})^p (A_{++})^q] |0\rangle_{FT}$ ($p, q \gg 1$). The total classical mass of the latter is $E = m_0(p + q)$, while they have J_+ , J_- charges p , $-q$. As a consequence, they satisfy the relation $E = m_0(J_+ - J_-) \equiv m_0 J$.

This is the value of the hadron energies in the limit of infinite λ , J , with $\lambda/J \rightarrow 0$. Specific string configurations with J_+ , J_- angular momenta, will provide specific informations on different “internal” structures of the above hadrons. Let us focus on the semiclassical circular solutions examined in the previous sections. The energy of the string ground state up

¹⁸For a discussion of similar states in general confining backgrounds, see [15].

¹⁹We use the same notation of [13], in which the second entry refers to the charge of the anti-fundamental representation, so “+” means charge $-1/2$ and “−” charge $+1/2$.

²⁰This is not calculable in perturbative field theory. Rather, it is a prediction of the string theory on the plane wave.

to one-loop order, in the limit of large J/λ is given by

$$E = m_0 J \left[1 + \frac{\lambda^2}{2J^2} \left(-m_1^2 \frac{J_-}{J} + m_2^2 \frac{J_+}{J} \right) + \dots \right] - m_0 J \left[\frac{Z^2(b)}{\lambda} + \dots \right], \quad (102)$$

where

$$2Z^2(b) \equiv \frac{b^2}{4} \log \frac{b^2}{4} - 2 \left(\frac{1}{2} + \frac{b}{4} \right)^2 \log \left(\frac{1}{2} + \frac{b}{4} \right)^2 - 2 \left(\frac{1}{2} - \frac{b}{4} \right)^2 \log \left(\frac{1}{2} - \frac{b}{4} \right)^2. \quad (103)$$

The first terms in (102) originate from the large J expansion of the classical energy (58), while the last ones account for the one-loop zero-point energy evaluated in the large k limit, i.e. formula (93). We see that these terms have different nature.

The terms coming from the classical energy are corrections in positive integer powers of the effective coupling λ^2/J^2 . Moreover they depend on the string winding numbers, which possibly account for some internal (spin-chain like) structure of the dual hadrons and which are remnant of the particular string solutions we are analyzing. The limit of large λ, J and small λ^2/J^2 we are taking can be read as an “effective perturbative” regime. Let us recall that $\lambda = Ne^{\Phi_0}$ is equal to the ratio between the string tension and the squared mass of the hadron constituents T_s/m_0^2 . So, we can speculate that taking the effective perturbative regime $T_s \ll Jm_0^2$ means considering a limit such that the contribution to the energy coming from the binding (which T_s accounts for) of the hadron constituents is much smaller than the one coming from the masses of the constituents.

The leading order corrections coming from the one-loop zero-point energy scale as $1/\lambda$ and as such they could be read as non perturbative (in λ) corrections to the mass of the single constituents of the hadrons.

Finally let us remember [12] that the low energy motion of our multicharged heavy states can be described by an effective nonrelativistic particle moving in the three special dimensions (corresponding to the massless modes δX^a).

7 Conclusions

Let us summarize the key results we have found in this paper. We analyzed folded and circular string solutions on the asymptotic of the (b)MN background (but our conclusions about the classical configurations apply to the (b)KS case as well) corresponding to the low energy regime of a confining $\mathcal{N} = 1$ (eventually softly broken) supersymmetric gauge theory coupled to an infinite tower of massive KK adjoint fields. String theory can be used to predict the energy/charge relations for hadrons formed by bound states of the latter. These relations can be easily studied in the limit of large charges, where a semiclassical analysis is possible. The classical string energy in the limit results to be proportional to the total charge, up to “regular” (i.e. positive powers in the effective coupling λ^2/J^2) corrections. Their presence is hard

to reproduce in the dual field theory, due to the fact that what we are really examining is the strong coupling regime of a confining gauge theory. Nevertheless a possible interpretation of these corrections, as due to the collective binding energy of the hadron, can be given.

The semiclassical analysis of the above solutions give more informations on the energy/charge relations. We studied the case of circular solutions, where we were able to obtain the small effective coupling expansion for the one-loop sigma model contribution E_1 to the energy. As a first result we found that the leading term in E_1 is the same that would have been obtained by considering the zero-point energy of the string on a generalized Penrose limit of the (b)MN background²¹. This term gives a correction to the classical energy/charge relation which scales as $1/\lambda$ and so is subleading with respect to the classical one (since we are assuming $\lambda \gg 1$). Nevertheless it is not suppressed in the limit of infinite total charge. This happens in general provided that the world-sheet supersymmetries are not linearly realized in the limit. This leading order quantum correction could be possibly interpreted as a non perturbative self renormalization of the mass of each single constituent of the dual hadron.

The main difference between the multispinning circular or folded string solutions and the point-like ones (which are semiclassically described by strings on pp-waves) is that the latter receive only the above kind of non perturbative, non analytic corrections to the energy/charge relation. At the classical level, in fact, string theory simply gives that E is proportional to J with no corrections to this relation. Thus they give different informations on the dual multiparticle bound states.

Let us stress that our results will generalize to all the regular backgrounds dual to confining gauge theories present in literature²². In fact in the IR regime their form is of the universal form $\mathbb{R}^{d,1} \times \mathbb{R}^q \times S^{9-d-q}$. The classical configurations are then the S^{9-d-q} -versions of the ones studied in [23, 24, 7] for $d + q = 4$. The quantization of the quadratic fluctuations can be performed along the lines of our section 4.

It would be very interesting to study the folded string solutions at the semiclassical level. These are likely to give stable configurations and it would be nice to understand the one-loop corrections, and in particular to study the sign of the zero-point energy.

Finally, let us compare our results with the $AdS_5 \times S^5$ ones. In the latter case, whenever the classical solution has a spin on S^5 , the expansion of the classical energy/charge relation is regular in λ_t/J^2 . Since a lot of these configurations have been successfully compared with field theory computations, the common believe is that this regularity implies the vanishing of the quantum corrections in the asymptotic $J \rightarrow \infty$ limit. In [35] it was claimed that this is due to the recovered supersymmetry (this was demonstrated for the circular strings on S^5) in the limit. Nevertheless, there are examples of asymptotically non-BPS solutions whose classical results agree with the field theory computations [36]. So, in [37] it was suggested that the

²¹In the analogous case of multispinning circular solutions on the S^5 of the $AdS_5 \times S^5$ background the same leading term is in fact zero (as the BMN zero-point energy).

²²See [11] for another example.

possibility of successfully perform such comparisons should depend on the fact that the strings become effectively tensionless in the $J \rightarrow \infty$ limit. The latter property is true because their size is bounded above by the radius of the five-sphere, so that the main contribution to the energy is the kinetic one.

The configurations we studied are non supersymmetric, even in the $J \rightarrow \infty$ limit. Of course, the classical bosonic solutions do not mind about supersymmetry. Since we have strings sitting at $\rho = 0$, they see, at the classical level, only the geometry $\mathbb{R}^{3,1} \times S^3$. The corresponding classical energy/charge relation is regular. This could be interpreted with the criterion above about the effective vanishing tension of these strings, since at this level the three-sphere radius is fixed. Note that the same criterion works for the other spinning strings on the MN background studied in [10]: the configurations extending in the radial direction (which is unbounded) do not exhibit regular energy/charge relations. Instead, the criterion seems to be untrue at the quantum level on general backgrounds.

In fact, for what concerns the quantum corrections, we have generically *non* subleading (in $1/J$) contributions in the $J \rightarrow \infty$ limit, while subleading ones for $b = 2$. As a first conclusion, we can say that on general backgrounds the regularity of the expansion of the classical energy/charge relation do *not* imply that the quantum contribution is $1/J$ suppressed. Since the latter implication is very likely to be true in the special case of $AdS_5 \times S^5$, it must depend on the specific form of this background. But, considering the example in [36], this property does not seem to be the huge amount of supersymmetry. Moreover, it cannot even be the fact that the radius of the five-sphere is bounded, because we have provided a counter-example: the $b = 2$ case has regular classical expansion and $1/J$ suppressed quantum corrections, but, as for the other values of b , the radius of the three-sphere is not fixed. In fact, at the quadratic fluctuation level needed for the calculation of the quantum corrections, the string does see that the three-sphere radius can increase²³. Note that this is visible already for the point-like strings of the pp-wave. This interesting subject surely requires more efforts for a complete understanding.

Acknowledgments

We thank F. Belgiorno, S. Cacciatori and T. Mateos for useful discussions. This work was partially supported by INFN, MURST and by the European Commission RTN program HPRN-CT-2000-00131.

A The GS superstring on a general background

In this appendix we review the form of the GS superstring action expanded up to the quadratic order (see [38] and references therein). The GS type IIB superstring action expanded up to the

²³If this would not have been the case, the $b \neq 2$ case would have provided a counter-example the other way around.

second order in the fermions is ($\xi^\alpha, \alpha = 0, 1$ are world-sheet coordinates, $x^m, m = 0, \dots, 9$ are the spacetime coordinates, $a, b, \dots = 0, \dots, 9$ are used for flat spacetime indexes, $\epsilon^{01} = 1$):

$$\begin{aligned}
S_{(F1)}^{IIB} &= -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-h} h^{\alpha\beta} \partial_\alpha x^m \partial_\beta x^n g_{mn} + \frac{1}{4\pi\alpha'} \int d^2\xi \epsilon^{\alpha\beta} \partial_\alpha x^m \partial_\beta x^n B_{mn} + \\
&\quad + \frac{i}{4\pi\alpha'} \int d^2\xi \sqrt{-h} \bar{\theta} (1 - \Gamma_{F1}) \Gamma^\alpha D_\alpha \theta + O(\theta^4) = \\
&= -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-h} h^{\alpha\beta} \partial_\alpha x^m \partial_\beta x^n g_{mn} + \frac{1}{4\pi\alpha'} \int d^2\xi \epsilon^{\alpha\beta} \partial_\alpha x^m \partial_\beta x^n B_{mn} + \\
&\quad + \frac{i}{4\pi\alpha'} \int d^2\xi [\sqrt{-h} h^{\alpha\beta} \delta^{IJ} - \epsilon^{\alpha\beta} (\sigma_3)^{IJ}] \bar{\theta}^I \Gamma_\alpha (D_J \theta)^J + O(\theta^4) .
\end{aligned} \tag{104}$$

We use the double spinor convention

$$\theta = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} , \tag{105}$$

where $\theta^{1,2}$ are MW spinors in ten dimensions with positive chirality ($\Gamma^{11}\theta^{1,2} = \theta^{1,2}$). Furthermore $\Gamma_\alpha = \partial_\alpha x^m e_m^a \Gamma_a$, the world-sheet chiral operator is

$$\Gamma_{F1} = \frac{1}{2\sqrt{-h}} \epsilon^{\alpha\beta} \Gamma_{\alpha\beta} \sigma_3 , \tag{106}$$

D_i is the pull-back on the world-sheet of the operator D_m defined by

$$D_m = D_m^{(0)} + W_m \tag{107}$$

with

$$\begin{aligned}
D_m^{(0)} &= \partial_m + \frac{1}{4} \omega_{mab} \Gamma^{ab} + \frac{1}{4 \cdot 2!} H_{mab} \Gamma^{ab} \sigma_3 , \\
W_m &= \frac{1}{8} e^\Phi \left[F_a^{(1)} \Gamma^a (i\sigma_2) + \frac{1}{3!} F_{abc}^{(3)} \Gamma^{abc} \sigma_1 + \frac{1}{2 \cdot 5!} F_{abcde}^{(5)} \Gamma^{abcde} (i\sigma_2) \right] \Gamma_m .
\end{aligned} \tag{108}$$

The operator D_m enters the gravitino supersymmetry transformation

$$\delta_\epsilon \psi_m = D_m \epsilon . \tag{109}$$

It is also useful to recall that the dilatino supersymmetry transformation takes the form

$$\delta_\epsilon \lambda = \Delta \epsilon , \tag{110}$$

where

$$\begin{aligned}
\Delta &= \Delta^{(1)} + \Delta^{(2)} , \\
\Delta^{(1)} &= \frac{1}{2} \left(\Gamma^m \partial_m \Phi + \frac{1}{2 \cdot 3!} H_{abc} \Gamma^{abc} \sigma_3 \right) ,
\end{aligned}$$

$$\Delta^{(2)} = -\frac{1}{2}e^\Phi \left[F_a^{(1)}\Gamma^a(i\sigma_2) + \frac{1}{2 \cdot 3!}F_{abc}^{(3)}\Gamma^{abc}\sigma_1 \right] . \quad (111)$$

In this case the κ -symmetry reads

$$\begin{aligned} \delta_\kappa \theta &= (1 - \Gamma_{F1})\kappa + O(\theta^2) , \\ \delta_\kappa x^m &= \frac{i}{2}\bar{\theta}\Gamma^m(1 - \Gamma_{F1})\kappa + O(\theta^3) . \end{aligned} \quad (112)$$

B UV finiteness and stability

In this appendix we discuss the large n limit of the frequencies for the quantized circular string configuration, showing that the theory is UV finite, and its stability. From the equation for the frequencies (80) one can derive the large n limit of the $\delta\chi$, $\delta\tilde{v}_1$ modes

$$\omega_b \sim n \pm B[(m_2^2 - m_1^2) + (w_1m_1 - w_2m_2)] - \frac{a_1^2m_1^2 + a_2^2m_2^2}{2n} + \frac{a_1^2w_1^2 + a_2^2w_2^2}{2n} . \quad (113)$$

The one for the δY^a modes read, from (82),

$$\begin{aligned} \omega_b &= n \pm \frac{1}{2}\sqrt{(M_2 + M_3)^2 + (M_4 + M_5)^2} + \\ &\quad + \frac{1}{n} \left[\frac{M}{2} + \frac{M_1}{2} - \frac{1}{8}(M_3^2 - M_2^2 + M_5^2 - M_4^2) - \frac{M_1(M_4 + M_5)^2}{4[(M_2 + M_3)^2 + (M_4 + M_5)^2]} \right] , \\ \omega_b &= n + \frac{1}{n} \left[\frac{M}{2} + \frac{M_1(M_4 + M_5)^2}{2[(M_2 + M_3)^2 + (M_4 + M_5)^2]} \right] . \end{aligned} \quad (114)$$

Taking into account the three δX^a modes the sum of the bosonic frequencies then give

$$\sum \omega_b = 8n + \frac{1}{n} \{ (a_1^2w_1^2 + a_2^2w_2^2) + \frac{b^2}{4} [2(a_1^2m_1^2 + a_2^2m_2^2) + (a_1^2w_1^2 + a_2^2w_2^2)] \} . \quad (115)$$

From the fermionic frequency equations (89) one derives the behavior

$$\omega_f \sim n \pm (C + D) + \frac{E^2 + F^2 + G^2 + H^2}{2n} . \quad (116)$$

Making explicit the coefficients one can check that the sum of these eight frequencies precisely cancels the bosonic sum, showing that the theory is indeed UV finite.

Finally, let us briefly discuss the stability of the theory. As in the AdS case, there exist some range of parameters where the frequencies become imaginary, signaling an instability of the solution. On the other hand, the important point is that this is not true in general. Let us begin from the frequencies for the $\delta\chi$, $\delta\tilde{v}_1$ bosons, which are obviously very similar to the sphere modes in the $AdS_5 \times S^5$ case. As a first observation, in the single charge limit

$m_2 = -m_1 \equiv m$, $w_2 = w_1 \equiv w$, $a_1 = a_2 = 1/\sqrt{2}$, the frequency equation (80) reduces to the analogous one in AdS and so gives the same stability condition $n^2 \geq 4m^2$ (cfr. (5.38) of [7]). But the interesting regime for our concern is the large $\nu^2 \equiv -\Lambda$ limit (or equivalently the large k limit), where (80) has a trivially safe mode $\omega_b \sim \alpha\nu$ with real α , and a second one going as (cfr. (5.41) of [7])

$$\omega_b \sim \frac{n}{\nu} [m_1 + m_2 + \frac{1}{2} \sqrt{n^2 - 4a_1^2 a_2^2 (m_1 - m_2)^2}] , \quad (117)$$

which gives the condition $n^2 \geq 4|m_1 m_2|$.

For what concerns the δY^a modes, one can show that there are values of the parameters where the frequency equation (82) have imaginary solutions. But in the large ν limit one finds two trivially safe modes and a third one going as $\omega_b \sim 1/\nu$ giving a non trivial, fourth order (in n), stability condition. A numerical analysis of the latter shows that the condition is always satisfied, so that in the limit the δY^a modes give no instabilities.

It is easy to observe from (89) that the fermionic frequencies are all safe, going as $\omega_f \sim \beta\nu$ with real coefficients β . In conclusion, the large ν regime has only the stability condition $n^2 \geq 4|m_1 m_2|$ as the two-charge circular solution in $AdS_5 \times S^5$. For generic ν other instabilities arise, but there is always a set of values of the other parameters for which the theory is safe.

References

- [1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113]; [arXiv:hep-th/9711200]. S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428 (1998) 105; [arXiv:hep-th/9802109]. E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253; [arXiv:hep-th/9802150].
- [2] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from $N = 4$ super Yang Mills,” JHEP 0204 (2002) 013; [arXiv:hep-th/0202021].
- [3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B 636 (2002) 99; [arXiv:hep-th/0204051].
- [4] R. Penrose, “Any space-time has a plane wave as a limit,” Differential geometry and relativity, Reidel, Dordrecht (1976) 271.
- [5] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory,” JHEP 0201 (2002) 047; [arXiv:hep-th/0110242]. M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “Penrose limits and maximal supersymmetry,” Class. Quant. Grav. 19 (2002) L87; [arXiv:hep-th/0201081].

- [6] R. R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background,” Nucl. Phys. B 625 (2002) 70; [arXiv:hep-th/0112044]. R. R. Metsaev and A. A. Tseytlin, “Exactly solvable model of superstring in plane wave Ramond-Ramond background,” Phys. Rev. D 65 (2002) 126004; [arXiv:hep-th/0202109].
- [7] A. A. Tseytlin, “Spinning strings and AdS/CFT duality,” arXiv:hep-th/0311139.
- [8] A. L. Larsen and A. Khan, “Novel explicit multi spin string solitons in AdS(5),” arXiv:hep-th/0312184. A. Ryang, “Folded Three-Spin String Solutions in AdS(5) x S(5),” arXiv:hep-th/0403180.
- [9] A. A. Tseytlin, “Semiclassical quantization of superstrings: AdS(5) x S(5) and beyond,” Int. J. Mod. Phys. A 18 (2003) 981; [arXiv:hep-th/0209116]. A. Armoni, J. L. F. Barbon and A. C. Petkou, “Rotating strings in confining AdS/CFT backgrounds,” JHEP 0210 (2002) 069; [arXiv:hep-th/0209224]. M. Alishahiha and A. E. Mosaffa, “Circular semiclassical string solutions on confining AdS/CFT backgrounds,” JHEP 0210 (2002) 060; [arXiv:hep-th/0210122]. S. A. Hartnoll and C. Nunez, “Rotating membranes on G(2) manifolds, logarithmic anomalous dimensions and $N = 1$ duality,” JHEP 0302 (2003) 049; [arXiv:hep-th/0210218]. M. Schvellinger, “Spinning and rotating strings for $N = 1$ SYM theory and brane constructions,” JHEP 0402 (2004) 066; [arXiv:hep-th/0309161]. M. Kruczenski, “Wilson loops and anomalous dimensions in cascading theories,” arXiv:hep-th/0310030.
- [10] J. M. Pons and P. Talavera, “Semi-classical string solutions for $N = 1$ SYM,” Nucl. Phys. B 665 (2003) 129; [arXiv:hep-th/0301178].
- [11] M. Alishahiha, A. E. Mosaffa and H. Yavartanoo, “Multi-spin string solutions in AdS black hole and confining backgrounds,” arXiv:hep-th/0402007.
- [12] E. G. Gimon, L. A. Pando Zayas, J. Sonnenschein and M. J. Strassler, “A soluble string theory of hadrons,” JHEP 0305 (2003) 039; [arXiv:hep-th/0212061].
- [13] R. Apreda, F. Bigazzi and A. L. Cotrone, “Strings on pp-waves and hadrons in (softly broken) $N = 1$ gauge theories,” JHEP 0312 (2003) 042; [arXiv:hep-th/0307055].
- [14] S. Kuperstein and J. Sonnenschein, “Analytic non-supersymmetric background dual of a confining gauge theory and the corresponding plane wave theory of hadrons,” JHEP 0402 (2004) 015; [arXiv:hep-th/0309011].
- [15] G. Bertoldi, F. Bigazzi, A. L. Cotrone, C. Nunez and L. A. Pando Zayas, “On the universality class of certain string theory hadrons,” arXiv:hep-th/0401031.
- [16] J. M. Maldacena and C. Nunez, “Towards the large N limit of pure $N = 1$ super Yang Mills,” Phys. Rev. Lett. 86 (2001) 588; [arXiv:hep-th/0008001].

- [17] S. S. Gubser, A. A. Tseytlin and M. S. Volkov, “Non-Abelian 4-d black holes, wrapped 5-branes, and their dual descriptions,” JHEP 0109 (2001) 017; [arXiv:hep-th/0108205].
- [18] F. Bigazzi, A.L. Cotrone and A. Zaffaroni, unpublished. N. Evans, M. Petrini and A. Zaffaroni, “The gravity dual of softly broken $N = 1$ super Yang-Mills,” JHEP 0206 (2002) 004; [arXiv:hep-th/0203203].
- [19] F. Bigazzi, A. L. Cotrone, M. Petrini and A. Zaffaroni, “Supergravity duals of supersymmetric four dimensional gauge theories,” Riv. Nuovo Cim. 25N12 (2002) 1; [arXiv:hep-th/0303191].
- [20] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities,” JHEP 0008 (2000) 052; [arXiv:hep-th/0007191].
- [21] V. Borokhov and S. S. Gubser, “Non-supersymmetric deformations of the dual of a confining gauge theory,” JHEP 0305 (2003) 034; [arXiv:hep-th/0206098].
- [22] R. Apreda, “Non supersymmetric regular solutions from wrapped and fractional branes,” arXiv:hep-th/0301118.
- [23] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, “Spinning strings in $AdS(5) \times S^{*5}$ and integrable systems,” Nucl. Phys. B 671 (2003) 3; [arXiv:hep-th/0307191].
- [24] G. Arutyunov, J. Russo and A. A. Tseytlin “Spinning strings in $AdS(5) \times S^{*5}$: New integrable system relations,” [arXiv:hep-th/0311004].
- [25] S. Frolov and A. A. Tseytlin, “Multi-spin string solutions in $AdS(5) \times S^{*5}$,” Nucl. Phys. B 668 (2003) 77; [arXiv:hep-th/0304255].
- [26] S. Frolov and A. A. Tseytlin, “Quantizing three-spin string solution in $AdS(5) \times S^{*5}$,” JHEP 0307 (2003) 016; [arXiv:hep-th/0306130].
- [27] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in $AdS(5) \times S(5)$,” JHEP 0206 (2002) 007; [arXiv:hep-th/0204226].
- [28] M. Bertolini and P. Merlatti, “A note on the dual of $N = 1$ super Yang-Mills theory,” Phys. Lett. B (2003) 80; [arXiv:hep-th/0211142].
- [29] L. A. Pando Zayas, J. Sonnenschein and D. Vaman, “Regge trajectories revisited in the gauge / string correspondence,” arXiv:hep-th/0311190.
- [30] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” Phys. Lett. B 570 (2003) 96; [arXiv:hep-th/0306143].

- [31] F. Bigazzi and A. L. Cotrone, “On zero-point energy, stability and Hagedorn behavior of type IIB strings on pp-waves,” JHEP 0308 (2003) 052; [arXiv:hep-th/0306102].
- [32] F. Bigazzi, A. L. Cotrone, L. Girardello and A. Zaffaroni, “pp-wave and non-supersymmetric gauge theory,” JHEP 0210 (2002) 030; [arXiv:hep-th/0205296].
- [33] C. Nunez, A. Paredes and A. V. Ramallo, “Flavoring the gravity dual of $N = 1$ Yang-Mills with probes,” JHEP 0312 (2003) 024; [arXiv:hep-th/0311201].
- [34] F. Bigazzi and A. L. Cotrone, “PP-waves and softly broken $N = 1$ SYM,” arXiv:hep-th/0401045.
- [35] D. Mateos, T. Mateos and P. K. Townsend, “Supersymmetry of tensionless rotating strings in $AdS(5) \times S^{*5}$, and nearly-BPS operators,” JHEP 0312 (2003) 017; [arXiv:hep-th/0309114].
- [36] J. Engquist, J. A. Minahan and K. Zarembo, “Yang-Mills duals for semiclassical strings on $AdS(5) \times S^{*5}$,” JHEP 0311 (2003) 063; [arXiv:hep-th/0310188].
- [37] D. Mateos, T. Mateos and P. K. Townsend, “More on supersymmetric tensionless rotating strings in $AdS(5) \times S^{*5}$,” arXiv:hep-th/0401058.
- [38] M. Cvetič, H. Lu, C. N. Pope and K. S. Stelle, “T-duality in the Green-Schwarz formalism, and the massless/massive IIA duality map,” Nucl. Phys. B 573 (2000) 149; [arXiv:hep-th/9907202]. L. Martucci and P. J. Silva, “On type II superstrings in bosonic backgrounds: the role of fermions and T-duality,” JHEP 0304 (2003) 004; [arXiv:hep-th/0303102].